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APPROXIMATE METHOD FOR ANALYSIS OF THE CONTACT TEMPERATURE AND PRESSURE DUE TO FRICTIONAL LOAD IN AN ELASTIC LAYERED MEDIUM

R. D. KULCHYTSKY-ZHYHAILO and A. A. YEVTUSHENKO
Department of Mechanics and Mathematics, Lviv State University, Universitetska Str. 1, 290602 Lviv, Ukraine

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Abstract—The present paper consists of two parts. The first part deals with the analysis of a temperature and the stress–strain state of the layered half-space by normal traction and is heated by heat flux distributed over the circle region of the top layer surface. The thermal and mechanical contact of the bodies is perfect. In the second part, the axisymmetric thermoelastic contact problem for the layered half-space with fractional heating is investigated. An approximate solution is obtained for a particular case in which the conductivity and rigidness of the thin layer is much less than that of the half-space. © 1997 Elsevier Science Ltd.

INTRODUCTION

The analysis of the stress state in layered media under combined mechanical and thermal loads is important for assessing mechanical reliability. Applications include protective coatings, layered composite media and thin films. The present paper gives the approximate solution of the thermoelastic contact problem involving frictional heating for a layered elastic medium. The Fourier integral transform technique on the space variable is utilized to solve this problem. The effect of different material properties on contact temperature and pressure distribution is given.

An axisymmetric contact problem has previously been reported for the case of a spherical indenter contacting an elastic half-space with no layer [see e.g. Barber (1975, 1976); Generalov *et al.* (1976); Grilitski and Kulchytsky-Zhyhailo (1991); Yevtushenko and Kulchytsky-Zhyhailo (1995, 1996)]. No analogous study has been performed for layered media. The isothermal normal contact problem for a single layer on a half-space has previously been studied by Keer *et al.* (1972, 1976), and Alexandrov and Mkhitaryan (1983).

SOLUTION OF THE THERMOELASTIC BOUNDARY PROBLEM FOR THE LAYERED HALF-SPACE

The problem considered here is that of an unbound layer in perfect contact with a semi-infinite base (half-space). The thickness of the layer is h and the rectangular coordinate system xyz is placed in relation to the two bodies, as shown in Fig. 1. The layer is pressed against the base by normal traction p and is heated by heat flux, q, distributed over the circle Ω, with radius a, on the top surface $z = h$ of the layer. The rest of this surface ($\bar{\Omega}$) is free and thermoinsulated.

In the dimensionless variables $\xi = x/a$, $\eta = y/a$, $\zeta = z/a$ the equilibrium equations can be expressed in terms of displacements $u^{(j)} = \{au_\xi^{(j)}, au_\eta^{(j)}, au_\zeta^{(j)}\}$ [see e.g. Nowacki (1986)]

$$(1-2\nu_j)\nabla^2 u^{(j)} + \nabla\,\mathrm{div}\,u^{(j)} = 2\alpha_j(1+\nu_j)a\nabla T^{(j)}, \quad j = 1,2, \tag{1}$$

where $T^{(j)}$ are the temperatures, ν_j are the Poisson's ratios, α_j are the thermal expansion

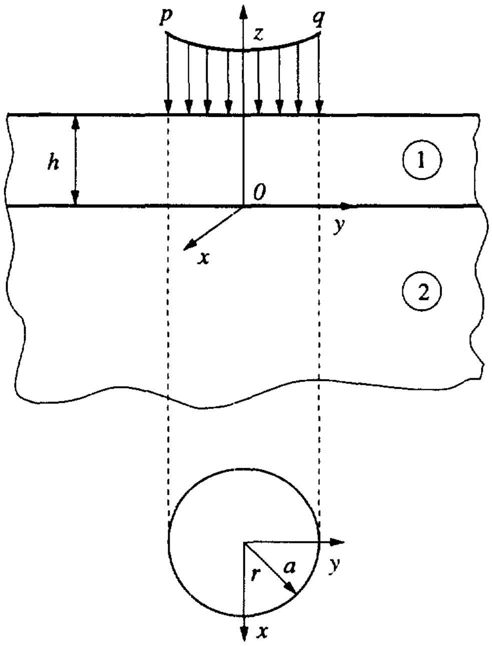

Fig. 1. Heated layer pressed against a half-space.

coefficients and $\nabla \equiv (\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\zeta)$. The subscripts and superscripts 1 and 2 refer to the layer and the base, respectively.

The heat conduction equations for these solids can be written in the form

$$\nabla^2 T^{(j)} = 0, \quad j = 1, 2. \tag{2}$$

The boundary conditions of the top surfaces $\zeta = h^* = h/a$ of the layer are

$$\sigma_{\zeta\zeta}^{(1)} = \begin{cases} -p(\xi,\eta) & \text{on } \Omega \\ 0 & \text{on } \bar{\Omega} \end{cases}, \tag{3}$$

$$\sigma_{\xi\zeta}^{(1)} = \sigma_{\eta\zeta}^{(1)} = 0 \quad \text{on } \Omega \bigcup \bar{\Omega}, \tag{4}$$

$$K_1 \frac{\partial T^{(1)}}{\partial \zeta} = \begin{cases} aq(\xi,\eta) & \text{on } \Omega \\ 0 & \text{on } \bar{\Omega} \end{cases}, \tag{5}$$

where k_j, $j = 1, 2$ are the thermal conductivity. The interface $\zeta = 0$ between the layer and the half-space is required to have continuous displacements, tractions, temperatures and heat fluxes for all values of ξ and η

$$u_\xi^{(1)} = u_\xi^{(2)}, \quad u_\eta^{(1)} = u_\eta^{(2)}, \quad u_\zeta^{(1)} = u_\zeta^{(2)}, \tag{6}$$

$$\sigma_{\zeta\zeta}^{(1)} = \sigma_{\zeta\zeta}^{(2)}, \quad \sigma_{\xi\zeta}^{(1)} = \sigma_{\xi\zeta}^{(2)}, \quad \sigma_{\eta\zeta}^{(1)} = \sigma_{\eta\zeta}^{(2)}, \tag{7}$$

$$T^{(1)} = T^{(2)}, \tag{8}$$

$$K_1 \frac{\partial T^{(1)}}{\partial \zeta} = K_2 \frac{\partial T^{(2)}}{\partial \zeta}. \tag{9}$$

We first seek a particular solution of the thermoelastic eqn (1) in the form [see e.g. Barber and Comninou (1989)]

$$u_\xi^{(j)} = \partial\varphi_j/\partial\xi, \quad u_\eta^{(j)} = \partial\varphi_j/\partial\eta, \quad u_\zeta^{(j)} = -\partial\varphi_j/\partial\zeta \tag{10}$$

where scalar potential functions φ_j, $j = 1, 2$, must satisfy the equations

$$\frac{\partial^2\varphi_j}{\partial\xi^2} + \frac{\partial^2\varphi_j}{\partial\eta^2} = \alpha_j(1+\nu_j)T^{(j)}, \quad \frac{\partial^2\varphi_j}{\partial\zeta^2} = -\alpha_j(1+\nu_j)T^{(j)}. \tag{11}$$

This solution is appropriate to the case of the half-space $\zeta > 0$ with no tractions on the surface $\zeta = 0$, i.e.

$$\sigma_{\zeta\zeta}^{(j)} = \sigma_{\xi\zeta}^{(j)} = \sigma_{\eta\zeta}^{(j)} = 0, \quad j = 1, 2. \tag{12}$$

The nonzero stress components are represented in a form analogous to Airy's stress function, but we emphasize that this is a true three-dimensional solution, since the functions φ_j and, hence, temperatures are a general three-dimensional harmonic function.

The homogeneous differential eqn (1) is reduced to three equations [see e.g. Galazyuk (1985)]

$$\nabla^2\theta^{(j)} = 0, \tag{13}$$

$$\nabla^2\chi^{(j)} = 0, \tag{14}$$

$$\nabla^2 u_\zeta^{(j)} = -d_j\partial\theta^{(j)}/\partial\zeta, \quad d_j = (1-2\nu_j)^{-1}, \tag{15}$$

where

$$\theta^{(j)} = \frac{\partial u_\xi^{(j)}}{\partial\xi} + \frac{\partial u_\eta^{(j)}}{\partial\eta} + \frac{\partial u_\zeta^{(j)}}{\partial\zeta}, \quad \chi^{(j)} = \frac{\partial u_\eta^{(j)}}{\partial\xi} - \frac{\partial u_\xi^{(j)}}{\partial\eta}. \tag{16}$$

Using condition (4) from eqn (14), we get $\chi^{(j)} = 0$. To solve the differential eqns (13) and (15), an integral Fourier transform technique with respect to the dimensionless coordinate variables ξ and η is used. Denoting the Fourier transformed quantities by a caret (ˆ), we obtain the analytic solutions of the eqns (13) and (15) in the transformed region as follows

$$\hat\theta^{(1)}(\alpha, \beta, \zeta) = C_1(\alpha, \beta)\cosh(s\zeta) + C_2(\alpha, \beta)\cosh[s(h^*-\zeta)] \tag{17}$$

$$\hat u_\zeta^{(1)}(\alpha, \beta, \zeta) = -1/2d_1C_1(\alpha, \beta)\zeta\cosh(s\zeta) + 1/2d_1C_2(\alpha, \beta)(h^*-\zeta)\cosh[s(h^*-\zeta)] + D_1(\alpha, \beta)\sinh(s\zeta) + D_2(\alpha, \beta)\sinh[s(h^*-\zeta)] \tag{18}$$

$$\hat\theta^{(2)}(\alpha, \beta, \zeta) = C_3(\alpha, \beta)\exp(s\zeta) \tag{19}$$

$$\hat u_\zeta^{(2)}(\alpha, \beta, \zeta) = -1/2d_2C_3(\alpha, \beta)\zeta\exp(s\zeta) + D_3(\alpha, \beta)\exp(s\zeta), \tag{20}$$

where

$$\begin{bmatrix}\hat{\theta}^{(j)}\\ \hat{u}_\zeta^{(j)}\end{bmatrix}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix}\theta^{(j)}\\ u_\zeta^{(j)}\end{bmatrix}\exp\left[-i(\alpha\xi+\beta\eta)\right]\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$s=\sqrt{\alpha^2+\beta^2},\quad i=\sqrt{-1}. \tag{21}$$

The components of displacements and stresses in the transformed space satisfy the relations

$$\hat{u}_\xi^{(j)}=\frac{i}{2\alpha}\left(\frac{\partial\hat{u}_\zeta^{(j)}}{\partial\zeta}-\hat{\theta}^{(j)}\right),\quad \hat{u}_\eta^{(j)}=\frac{i}{2\beta}\left(\frac{\partial\hat{u}_\zeta^{(j)}}{\partial\zeta}-\hat{\theta}^{(j)}\right) \tag{22}$$

$$\hat{\sigma}_{\zeta\zeta}^{(j)}=2\mu_j\left(2\frac{\partial\hat{u}_\zeta^{(j)}}{\partial\zeta}+(d_j-1)\hat{\theta}^{(j)}\right) \tag{23}$$

$$\hat{\sigma}_{\xi\zeta}^{(j)}=\frac{i}{\alpha}\mu_j\left((s^2+2\alpha^2)\hat{u}_\zeta^{(j)}-(1+d_j)\frac{\partial\hat{\theta}^{(j)}}{\partial\zeta}\right) \tag{24}$$

$$\hat{\sigma}_{\eta\zeta}^{(j)}=\frac{i}{\beta}\mu_j\left((s^2+2\beta^2)\hat{u}_\zeta^{(j)}-(1+d_j)\frac{\partial\hat{\theta}^{(j)}}{\partial\zeta}\right) \tag{25}$$

where μ_j are the shear moduli.

Substituting the transformed boundary condition (3) and the matching conditions (6) and (7) to the solutions (17)–(20) and taking relations (22)–(25) into account, we obtain the six algebraic equations regarding six coefficients C_k, D_k, $k=1,2,3$ (see Appendix).

From the solution of the heat conductivity problem (2), (8) and (9) we find the Fourier transform of the temperatures $T^{(j)}$ in the form:

$$\hat{T}^{(1)}(\alpha,\beta,\zeta)=\frac{\hat{q}(\alpha,\beta)a}{K_1 s\sinh(sh^*)}\left\{\cosh(s\zeta)-\frac{\cosh[s(h^*-\zeta)]}{\cosh(sh^*)+K^*\sinh(sh^*)}\right\}$$

$$\hat{T}^{(2)}(\alpha,\beta,\zeta)=\frac{\hat{q}(\alpha,\beta)a\exp(s\zeta)}{K_2 s[\cosh(sh^*)+K^*\sinh(sh^*)]}, \tag{26}$$

where $K^*=K_1/K_2$.

At $\zeta=h^*$, from eqns (10), (11) and (18), the following result is established

$$\hat{u}_\zeta^{(1)}(\alpha,\beta,h^*)=\frac{1+d_1}{2s}C_1(\alpha,\beta)\sinh(sh^*)+\frac{\alpha_1(1+\nu_1)}{s^2}\left.\frac{\partial\hat{T}^{(1)}}{\partial\zeta}\right|_{\zeta=h^*}, \tag{27}$$

where the functions C_1 and $\partial\hat{T}^{(1)}/\partial\zeta$ are still given by eqns (A5) and (26), respectively.

The expression for the transformed surface temperature is

$$\hat{T}^{(1)}(\alpha,\beta,h^*)=\frac{\hat{q}(\alpha,\beta)a}{K_2 s}+\frac{a}{K_2 s}\left(\frac{1}{K^*}-K^*\right)\frac{\hat{q}(\alpha,\beta)\sinh(sh^*)}{\cosh(sh^*)+K^*\sinh(sh^*)}. \tag{28}$$

We note that in the case of the corresponding isothermal problem ($q=0$) at $\mu^*\to 0$, from eqns (A5) and (27), we obtain a well-known result for the normal surface displacement of the elastic layer perfectly adhering to a rigid foundation due to a normal traction p [see e.g. Vorovich *et al.* (1974)].

If the thickness h^* is very small ($h^*\ll 1$), then the asymptotic analysis of the eqns (27) and (28) by the Kovalenko (1990) technique (i.e. including the terms $0(h^*)$ only) leads to

$$u_\zeta^{(1)}(\xi,\eta,h^*) \approx -\frac{1-\nu_2}{4\pi^2\mu_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\sqrt{\alpha^2+\beta^2}}\,\mathrm{d}\alpha\,\mathrm{d}\beta\iint_\Omega p(x,y)$$

$$\times\exp[-i(\alpha x+\beta y)]\,\mathrm{d}x\,\mathrm{d}y+\frac{\delta_2 a}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\alpha^2+\beta^2}\,\mathrm{d}\alpha\,\mathrm{d}\beta\iint_\Omega q(x,y)$$

$$\times\exp[-i(\alpha x+\beta y)]\,\mathrm{d}x\,\mathrm{d}y-\frac{(d_2+\mu^*)(d_2-d_1\mu^*)}{\mu_1(1+d_1)d_2^2}h^*p(\xi,\eta)$$

$$+\left[\frac{2d_1\mu^*-d_2+d_1d_2}{(1+d_1)d_2}\left(\delta_1-\frac{\delta_2}{K^*}\right)+\delta_1-\delta_2\right]K_1T^{(1)}(\xi,\eta,h^*) \tag{29}$$

$$T^{(1)}(\xi,\eta,h^*)=\frac{a}{4\pi^2K_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\sqrt{\alpha^2+\beta^2}}\,\mathrm{d}\alpha\,\mathrm{d}\beta\iint_\Omega q(x,y)\exp[-i(\alpha x+\beta y)]\,\mathrm{d}x\,\mathrm{d}y$$

$$+\frac{a}{K_2}\left(\frac{1}{K^*}-K^*\right)h^*q(\xi,\eta), \tag{30}$$

where $\delta_i=\alpha_i(1+\nu_i)/K_i$ are the thermal distortivities.

We observed that the last two terms in eqn (29) and the last term in eqn (30) determine the effect of the thin elastic conducting layer (film). It is clear that at $h^*\to 0$ this equation contains the equation for the half-space.

The question is, for which relative thickness of the layer can eqns (29) and (30) be used? To give the exact formulated answer is not simple. To get an approximate estimation of it, the solution of the following assistant problem of the steady heat conductivity can be investigated. The constant heat flux q is applied to the top surface of the layer in circle with the radius a. The exact solution of this axisymmetric problem at $\zeta=h^*$ is

$$T^{(1)}(\rho,h^*)=\frac{aq}{K_2}\int_0^\infty s^{-1}J_1(s)J_0(s\rho)\left[1+\left(\frac{1}{K^*}-K^*\right)\times\frac{\sinh(sh^*)}{\cosh(sh^*)+K^*\sinh(sh^*)}\right]\mathrm{d}s \tag{31}$$

and the approximate solution is

$$T^{(1)}(\rho,h^*)=\frac{aq}{K_2}\left[\int_0^\infty s^{-1}J_1(s)J_0(s\rho)\,\mathrm{d}s+\left(\frac{1}{K^*}-K^*\right)h^*\right], \tag{32}$$

where $\rho=\sqrt{\xi^2+\eta^2}$, $J_0(\cdot)$, $J_1(\cdot)$ are the Bessel functions of the first kind. It follows immediately that if

$$\frac{1}{h^*}\int_0^\infty s^{-1}J_1(s)J_0(s\rho)\frac{\sinh(sh^*)}{\cosh(sh^*)+K^*\sinh(sh^*)}\,\mathrm{d}s\approx 1,\quad \rho<1, \tag{33}$$

then the solutions (31) and (32) perfectly coincide. Hence, eqn (33) is a criterion for such a h^* definition when the suggested approximate approach can be used. The numerical analysis shows that the value h^*, at which eqn (33) takes place, essentially depends on the parameter K^*. So, at $K^*=0.1$ we have $h^*<0.2$, at $K^*=1$, $h^*<0.05$, at $K^*=5$, $h^*<0.01$. This can be explained by the fact that the coefficient $(sh^*)^2$ in the series expansion of $\hat{T}^{(1)}(s,h^*)$, by the powers sh^*, increases with increasing K^*.

By a certain analogy between eqns (27) and (28) we find that the influence of the parameters K^* and μ^* on the solution accuracy is rather similar. Thus, the suggested approach works well in the case when the conductivity and rigidness of the thin layer is much less than that of the substratum.

CONTACT PROBLEM WITH FRICTIONAL HEAT GENERATION

Let us consider two sliding semi-infinite solids which are pressed together by force, P, and nominally conform over some contact area. The mechanical energy is associated with friction losses and transfers into heat. The generation of heat at the contact region, due to friction, will induce temperature gradients and thermoelastic distortion and, hence, affect the contact pressure distribution.

Thermoelastic contact problems of this type are extremely intractable, since they involve moving heat sources, combined normal and tangential loading of the solids and a contact area whose dimensions are unknown *a priori* [see e.g. Barber (1976)]. However, yet not without significance, the problem may be somewhat simplified by the following assumptions:

(a) the contact area is stationary with respect to one, layered solid, in which a steady flow of heat is directed. the composite medium consists of an elastic layer of thickness h and the semi-space as shown in Fig. 1;

(b) the other solid is a rigid non-conductor;

(c) the coupling between tangential and normal traction in the contact area can be neglected;

(d) the conducting layered body is slightly rounded with a radius of curvature R to give the contact circle an area with radius $a \ll R$;

(e) the coefficient of friction f is constant throughout the contact area.

Some remarks should be made about the assumption (c). This assumption does not mean that the tangential traction on the surface is neglected. Indeed, the work done against these tractions is the source of the heat generation. However, the normal elastic displacements, caused by the tangential tractions, are much smaller than those produced by the normal tractions, and the coupling effect is negligible [see e.g. Barber and Comninou (1989)]. Hence, the boundary value problem for determining the contact pressure distribution is the same as that which would be generated if the contact was frictionless, except for the inclusion of the appropriate heat input.

Under these assumptions, the problem will be axisymmetrical. The heat input q in the cylindrical coordinate system (r, z) to the thermally conducting layered solid is equal to the rate of the frictional heat generation throughout the contact area and in the separation region there is no heat flux across the surface. Hence

$$q(\rho) = \begin{cases} fvp(\rho), & \rho \leqslant 1 \\ 0, & \rho > 1 \end{cases} \tag{34}$$

where v is the sliding speed, $\rho = r/a$.

For the case of very soft layer (where the layer modulus of rigidity is much smaller than the substratum) and when the substratum is thermally undeformed, from eqn (29) at $\mu^* \to 0$ and $\alpha_2 \to 0$ we find

$$u_\zeta^{(1)}(\rho, h^*) = -\frac{h^*}{\mu_1(1+d_1)}p(\rho) + \frac{2d_1\alpha_1(1+v_1)h^*}{1+d_1}T^{(1)}(\rho, h^*), \quad \rho \geqslant 0. \tag{35}$$

The contact temperature from eqn (30) is given by

$$T^{(1)}(\rho, h^*) = \frac{a}{K_2}\left(\frac{1}{K^*} - K^*\right)h^*q(\rho) + \frac{a}{K_2}\int_0^\infty J_0(s\rho)\,\mathrm{d}s \int_0^1 \tau q(\tau) J_0(\tau s)\,\mathrm{d}\tau, \quad \rho \geqslant 0. \tag{36}$$

Substituting for $u_\zeta^{(1)}$ (35) into mechanical boundary condition

$$u_\xi^{(1)} = \frac{\rho^2 a}{2R} - \Delta, \quad \rho \leqslant 1 \tag{37}$$

(Δ is the rigid displacement of layer body) and taking eqns (34) and (40) into account, we obtain the Fredholm integral equation

$$p(\rho) = \beta^* a \int_0^\infty J_0(s\rho)\,\mathrm{d}s \int_0^1 \tau p(\tau) J_0(\tau s)\,\mathrm{d}\tau + \frac{\mu_1(1+d_1)}{h^*}\left(\Delta - \frac{a\rho^2}{2R}\right), \quad 0 \leqslant \rho \leqslant 1, \tag{38}$$

where

$$\beta^* = \frac{\delta_1 f v K^*}{\gamma}, \quad \gamma = \frac{1-2v_1}{2\mu_1}. \tag{39}$$

The load P is given by

$$2\pi a^2 \int_0^1 \rho p(\rho)\,\mathrm{d}\rho = P. \tag{40}$$

We can use the continuous continuity $p(1) = 0$ to determine the unknown rigid-body displacement term Δ, in eqn (38) obtaining

$$\frac{\mu_1(1+d_1)}{h^*}\Delta = \frac{\mu_1(1+d_1)a}{2h^* R} - \beta^* a \int_0^\infty J_0(s)\,\mathrm{d}s \int_0^1 \tau p(\tau) J_0(\tau s)\,\mathrm{d}\tau. \tag{41}$$

Substituting eqn (41) into eqn (38), we obtain the integral equation

$$p(\rho) = \frac{\mu_1(1+d_1)a}{2h^* R}(1-\rho^2) + \beta^* a \int_0^\infty [J_0(s\rho) - J_0(s)]\,\mathrm{d}s \int_0^1 \tau p(\tau) J_0(\tau s)\,\mathrm{d}\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{42}$$

The limit value of the input parameter $\beta^* a$ at which the Fredholm integral eqn (42) has a unique solution we find by Yevtushenko and Kulchytsky-Zhyhailo (1995) method

$$\beta^* a < 2.64, \tag{43}$$

where we obtain the critical value of the contact area radius

$$a_{\mathrm{cr}} = \frac{2.64\gamma}{\delta_1 f v K^*}. \tag{44}$$

To obtain an approximate solution (42), we represent $p(\rho)$ in the form

$$p(\rho) = \frac{\mu_1(1+d_1)a}{2h^* R}(1-\rho^2)p^*(\rho), \tag{45}$$

where $p^*(\rho)$ is the unknown bounded function. We now substitute eqn (45) into eqn (42) to find

$$p^*(\rho) = 1+\beta^* a \int_0^\infty \frac{J_0(s\rho)-J_0(s)}{1-\rho^2} \mathrm{d}s \int_0^1 \tau p^*(\tau)(1-\tau^2) J_0(\tau s)\,\mathrm{d}\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{46}$$

The integral equation (46) is discretized by replacing the actual distribution of pressure by a piece-wise constant representation. We divide the interval [0, 1] into N parts by points $a_m = m/N$, $m = 0, 1, \ldots, N$ and assume that $p^*(\rho) = p_m^* = \text{const}$ at $a_{m-1} \leqslant \rho \leqslant a_m$. After integrating we obtain the system of linear algebraic equations

$$\sum_{m=1}^{N} b_{km} p_m^* = 1, \quad k = 1, 2, \ldots, N, \tag{47}$$

where

$$b_{km} = \delta_{km} - \beta^* a (1-\rho_k^2)^{-1} [B(\rho_k, a_m) - B(\rho_k, a_{m-1})]$$

$$B(\rho, a) = \begin{cases} a(1-a^2)F(1/2, -1/2; 1; \rho^2/a^2) + 2/3a^3 \\ \quad \times F(1/2, -3/2; 1; \rho^2/a^2) - 1/2a^2(1-a^2) \\ \quad \times F(1/2, 1/2; 2; a^2) - 1/4a^4 F(1/2, 1/2; 3; a^2), & \rho < a \\ 1/2a^2(1-a^2)\rho^{-1} F(1/2, 1/2; 2; a^2/\rho^2) + 1/4a^4 \\ \quad \times \rho^{-1} F(1/2, 1/2; 3; a^2/\rho^2) - 1/2a^2(1-a^2) \\ \quad \times F(1/2, 1/2; 2; a^2) - 1/4a^4 F(1/2, 1/2; 3; a^2), & \rho > a \end{cases} \tag{48}$$

$\rho_k = (k-1/2)/N$, $k = 1, 2, \ldots, N$, δ_{km} is the Kronecker delta, F is the Gauss hypergeometric function [see e.g. Abramowitz and Stegun (1964)].

The dependence of P_H/P (P_H is the load required to maintain the contact radius at a fixed value a in the corresponding isothermal Hertz contact problem) on the parameter $\beta^* a$ was investigated. It was found that this dependence is linear in accordance with the equation

$$\frac{P_H}{P} = 1 - \frac{\beta^* a}{2.64}. \tag{49}$$

The distribution of the dimensionless contact pressure $p(\rho)/p(0)$ for several values of the parameter $\beta^* a$ is shown in Fig. 2.

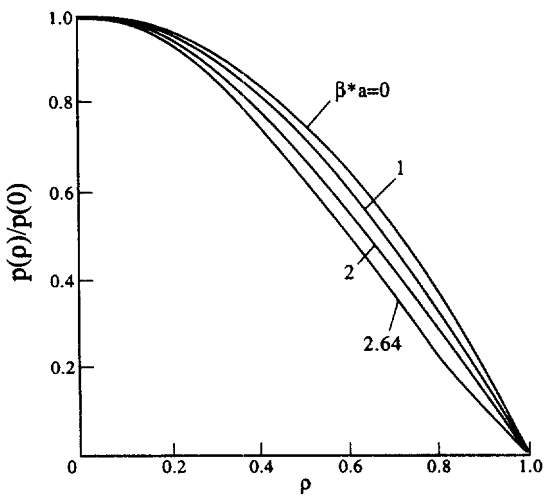

Fig. 2. Distribution of the dimensionless contact pressure $p(\rho)/p(0)$.

We now consider the case in which the layered solid surface is plane, i.e. $1/R = 0$ and the contact occurs over a circle of the fixed radius a. By defining

$$p(\rho) = \frac{\mu_1(1+d_1)\Delta}{h^*} p^*(\rho), \tag{50}$$

where $p^*(\rho)$ is the unknown function, from eqn (38) to obtain

$$p^*(\rho) = 1+\beta^* a \int_0^\infty J_0(s\rho)\,\mathrm{d}s \int_0^1 \tau p^*(\tau) J_0(\tau s)\,\mathrm{d}\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{51}$$

We can, therefore, use eqn (40) to determine the unknown rigid-body displacement term Δ in the form

$$\Delta = \frac{P_1}{P^*}, \quad P_1 = \frac{Ph^*}{2\pi a^2 \mu_1(1+d_1)}, \quad P^* = \int_0^1 \rho p^*(\rho)\,\mathrm{d}\rho, \tag{52}$$

where $p^*(\rho)$ is the solution of the integral equation (51).

As is seen from eqn (52), P_1 is a fixed parameter independent of the solution of eqn (51) and P^* is determined by the solution $p^*(\rho)$ of eqn (51), dependent on the parameter $\beta^* a$.

The numerical solution of eqn (51) is obtained by means of the algorithm previously described. As a result we get the following conclusions:

(1) if $0 \leqslant \beta^* a < 1.12$, then $P^* \geqslant 0$ and $P^* \to \infty$ at $\beta^* a \to 1.12$. Thus, one obtains $\Delta > 0$. This means that the mechanical deformations exceed the thermal ones, therefore, with increasing the force P the amount Δ also increases;

(2) if $\beta^* a > 1.12$, then $\Delta < 0$. In this case thermal deformations exceed mechanical ones. This means that with an increase of the load, P, the rigid solid rises up more. The physical sense of the solution ($p(\rho) \geqslant 0$) remains at $\beta^* a < 2.64$. At $\beta^* a > 2.64$, one realizes that the cylinder comes out of the layer surface. In this case, one finds the contact pressure as the solution to the uniform equation

$$p(\rho) = 2.64 \int_0^\infty [J_0(s\rho) - J_0(s)]\,\mathrm{d}s \int_0^1 \tau p(\tau) J_0(\tau s)\,\mathrm{d}\tau, \quad 0 \leqslant \rho \leqslant 1 \tag{53}$$

at the additional condition (40).

The existence of the nontrivial solution to integral eqn (53) is satisfied by the coincidence of the number 2.64 with the first real eigenvalue of the corresponding integral compact operator. As a consequence, one realizes that the position of the rigid–solid point, coming out of the layer surface, is not dependent on a load.

CONCLUSION

The solution of steady-state axisymmetric thermoelastic problem for a composite medium is obtained which is consistent of a finite thickness layer and the half-space. When the layer thickness is sufficiently small (a film), it was established that its behaviour is analogous to Winkler's elastic foundation:

(1) the elastic part of a normal surface displacement is proportional to the contact pressure, and the thermal part is proportional to the contact temperature [equation (20)];

(2) the contact temperature is proportional to the heat flow [eqn (30)].

The greatest layer thickness, at which takes place the above inferences, is determined by parameters μ^* and K^*. When decreasing these parameters, the error caused by using eqns (29) and (30) also decreases. Thus, the proposed approximate method is rather effective for composite media whose rigidity and conductivity of the surface film is much less than the corresponding semi-space parameters.

As an example, one considers a contact problem with frictional heating for layered conductivity solid and a rigid non-conductor punch. Two cases of interaction are studied with an unknown *a priori* contact region and a fixed contact area. In the first case, it was established that a critical value of the contact circle radius exists when the total unbound load increases. In the second case, one studies the interaction between mechanical and thermal deformation within the contact region. It is shown that at a certain value of the independent input parameter β^*a, the detachment of the rigid solid from the film surface occurs. The suggested approximated technique can also be applied to the investigation of some other mixed-value problems of steady-state thermoelasticity.

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APPENDIX

The governing system of the linear algebraic equations for the coefficiences C_k, D_k, $k = 1, 2, 3$ is

$$[A](C_1, C_2, C_3, D_1, D_2, D_3) = (0, \bar{p}/\mu_1, F_1, F_2, 0, 0), \tag{A1}$$

where the nonzero components of the matrix $\mathbf{A} = (a_{mn})$ are

$$\begin{aligned}
a_{11} &= (1+d_1)\sinh(sh^*) + d_1 sh^* \cosh(sh^*), a_{14} = -2s\sinh(sh^*),\\
a_{21} &= \cosh(sh^*) + d_1 sh^* \sinh(sh^*), a_{22} = 1, a_{24} = -2s\cosh(sh^*),\\
a_{25} &= 2s, a_{32} = -1/2d_1 h^* \cosh(sh^*), a_{35} = -\sinh(sh^*), a_{36} = 1,\\
a_{41} &= 1 + 1/2d_1, a_{42} = (1+1/2d_1)\cosh(sh^*) + 1/2d_1 sh^* \sinh(sh^*),\\
a_{43} &= -(1+1/2d_2), a_{44} = -s, a_{45} = s\cosh(sh^*), a_{46} = s,\\
a_{51} &= -\mu_1, a_{52} = -\mu_1[\cosh(sh^*) + d_1 sh^* \sinh(sh^*)], a_{53} = \mu_2,\\
a_{54} &= 2\mu_1 s, a_{55} = -2\mu_1 s\cosh(sh^*), a_{56} = -2\mu_2 s,\\
a_{62} &= \mu_1[(1+d_1)\sinh(sh^*) + d_1 sh^* \cosh(sh^*)],\\
a_{63} &= \mu_2(1+d_2), a_{65} = 2\mu_1 s\sinh(sh^*), a_{66} = -2\mu_2 s,
\end{aligned} \tag{A2}$$

and the functions F_i, $i = 1, 2$ are given by

$$F_1(\alpha,\beta) = -\frac{1}{s^2}\sum_{j=1}^{2}(-1)^j\alpha_j(1+\nu_j)\frac{\partial \hat{T}^{(j)}(\alpha,\beta,\zeta)}{\partial\zeta}\bigg|_{\zeta=0} \tag{A3}$$

$$F_2(\alpha,\beta) = \sum_{j=1}^{2}(-1)^j\alpha_j(1+\nu_j)\hat{T}^{(j)}(\alpha,\beta,\zeta)\bigg|_{\zeta=0}. \tag{A4}$$

The analytical solution of the algebraic eqn (A1) is

$$C_1(\alpha,\beta) = C_1^*(\alpha,\beta)/C(\alpha,\beta), \quad C_2(\alpha,\beta) = C_2^*(\alpha,\beta)/C(\alpha,\beta),$$

$$\begin{aligned}
C_1^*(\alpha,\beta) = {} & -\frac{\hat{p}(\alpha,\beta)}{\mu_1}[\mu^*(1+d_1)(1+d_2)\cosh(sh^*)+d_1(\mu^*-1) \\
& \times(d_2+2\mu^*+d_2\mu^*)sh^*+(2\mu^*+2d_2+2d_1\mu^*+d_1d_2(1+\mu^{*2})) \\
& \times\sinh(sh^*)\cosh(sh^*)]-2sF_1(\alpha,\beta)[d_2(1+d_1)\cosh(sh^*) \\
& +\mu^*d_1(1+d_2)[\sinh(sh^*)+sh^*\cosh(sh^*)]+d_1(\mu^*+d_2)sh^*\sinh(sh^*)] \\
& -2F_2(\alpha,\beta)[\mu^*d_1(1+d_2)sh^*\sinh(sh^*)-(d_1\mu^*-d_2)\sinh(sh^*) \\
& +d_1(\mu^*+d_2)sh^*\cosh(sh^*)]
\end{aligned}$$

$$\begin{aligned}
C_2^*(\alpha,\beta) = {} & \frac{\hat{p}(\alpha,\beta)}{\mu_1}[(\mu^*+d_2+d_1(2+d_2\mu^{*2})-\mu_1^*d_1)\sinh(sh^*) \\
& +\mu^*(1+d_1+d_2+d_1d_2)\cosh(sh^*)+d_1(2\mu^*+d_1+d_2\mu^*)(\mu^*-1) \\
& \times sh^*\cosh(sh^*)]+2sF_1(\alpha,\beta)[d_1d_2\cosh^2(sh^*)+d_1\mu^*\sinh^2(sh^*) \\
& +d_1(1+d_2)\mu^*(\sinh(sh^*)\cosh(sh^*)+sh^*)+d_2]+2F_2(\alpha,\beta) \\
& \times[d_1(1+d_2)\mu^*\sinh^2(sh^*)+d_1(\mu^*+d_2)(\sinh(sh^*)\cosh(sh^*)+sh^*)]
\end{aligned}$$

$$\begin{aligned}
C(\alpha,\beta) = {} & \sinh(sh^*)[d_1(\mu^*+2d_2+d_1d_2)\cosh(sh^*)+d_1\mu^*(1+2\mu^*d_1 \\
& +\mu^*d_1d_2)\sinh^2(sh^*)+2d_1\mu^*(1+d_1)(1+d_2)\sinh(sh^*)\cosh(sh^*) \\
& -d_1^2(\mu^*-1)(d_2+2\mu^*+d_2\mu^*)(sh^*)^2+(d_2-d_1\mu^*)]
\end{aligned}$$

$$\begin{aligned}
C_3(\alpha,\beta) = {} & (2+d_2)^{-1}\{(1+d_1sh^*+d_1\cosh^2(sh^*)+d_1\sinh(sh^*) \\
& \times\cosh(sh^*))C_1(\alpha,\beta)+((1+d_1)\cosh(sh^*)-\sinh(sh^*) \\
& +d_1sh^*[\sinh(sh^*)+\cosh(sh^*)])C_2(\alpha,\beta)+\mu_1^{-1}[\sinh(sh^*) \\
& +\cosh(sh^*))\hat{p}(\alpha,\beta)+2sF_1(\alpha,\beta)-2F_2(\alpha,\beta)\}
\end{aligned}$$

$$D_1(\alpha,\beta) = \frac{(1+d_1)\sinh(sh^*)+d_1sh^*\cosh(sh^*)}{2s\sinh(sh^*)}C_1(\alpha,\beta)$$

$$D_2(\alpha,\beta) = \frac{d_1[\sinh(sh^*)\cosh(sh^*)+sh^*]}{2s\sinh(sh^*)}C_1(\alpha,\beta)-\frac{C_2(\alpha,\beta)}{2s}+\frac{\hat{p}(\alpha,\beta)}{2\mu_1 s}$$

$$\begin{aligned}
D_3(\alpha,\beta) = {} & \frac{d_1[\sinh(sh^*)\cosh(sh^*)+sh^*]}{2s}C_1(\alpha,\beta)+C_2(\alpha,\beta) \\
& \times\frac{d_1sh^*\cosh(sh^*)-\sinh(sh^*)}{2s}+\frac{\sinh(sh^*)}{2\mu_1 s}\hat{p}(\alpha,\beta)+F_1(\alpha,\beta).
\end{aligned} \tag{A5}$$

Here

$$\mu^* = \mu_1/\mu_2. \tag{A6}$$